J. Math. Anal. Appl. 382 (2011) 856–863

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the domain dependence of the inf–sup and related constants via conformal mapping

Sándor Zsuppán

Berzsenyi Dániel Evangélikus Gimnázium, H-9400 Sopron, Széchenyi tér 11, Hungary

ARTICLE INFO

Article history:
Received 17 February 2011
Available online 4 May 2011
Submitted by D. Khavinson

Keywords:
Inf–sup constant
Friedrichs' inequality
Conformal mapping

ABSTRACT

In this paper we investigate the domain dependence of the inf–sup stability constant in the family of two-dimensional simply connected domains using its connection to the optimal constant figuring in Friedrichs' inequality for conjugate harmonic functions and the conformal mapping of the domain. A lower estimation of the inf–sup constant is also given in terms of the conformal mapping provided the boundary of the domain is smooth enough. We illustrate the results with several examples.

1. Introduction

The first kind Stokes problem describes the slow motion of a viscous incompressible fluid in the domain Ω driven by external forces. The inf–sup condition

$$\inf_{0\neq p\in L_{2,0}(\Omega)} \sup_{0\neq\vec{u}\in(W_0^{1,2}(\Omega))^2} \frac{(\operatorname{div}\vec{u}, p)_0^2}{(\vec{u},\vec{u})_1(p,p)_0} = \beta_0^2(\Omega) > 0, \tag{1}$$

is the most important inequality in the theory of the Stokes flow. It guarantees the stable solvability and uniqueness of the solutions of the first kind Stokes problem for the velocity and the pressure functions on the domain [5,21]. It is also important for the iterative solution of discretized Stokes and Navier–Stokes problems [2,10]. The inf–sup constant $\beta_0(\Omega)$ figuring therein is important for stability and error estimates for the solutions of the Stokes problem. Regardless of its importance, explicit values of the inf–sup constant for domains are known only in a few cases: for the circle, the annulus [3], and the ellipse, see [12], and for an infinite strip – assuming periodicity along the strip [14], and, in the three-dimensional case, for the sphere [22]. For plane domains which are the images of the unit disc by a polynomial conformal mapping the inf–sup constant can be calculated as an eigenvalue of a certain finite matrix, see [24,25]. Such domains are in a special class of plane domains, called quadrature domains [17]. Some estimations for the inf–sup constant are derived in [4,6,20].

The inf–sup constant of a simply connected plane domain with Lipschitz boundary is connected to several other domain specific optimal constants figuring in important inequalities [12,19,22]. The most important for the aim of the present paper is the connection with Friedrichs' inequality [7] for the real and imaginary part of a square integrable analytic function provided the real part has zero integral mean or it has prescribed zero value in a fixed point. Another related constant is that in Korn's second inequality [8,11] and that in the Babuška–Aziz inequality [1,15,23]. The connection between the inf–sup and Friedrichs' constants for multiple connected plane domains is shown implicitly in [9] assuming that the boundary of the domain has continuous curvature.

E-mail address: zsuppan@emk.nyme.hu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.04.086

The primary aim of this paper is to compare the inf–sup constants of two simply connected plane domain in terms of their corresponding conformal mappings.

In Section 2 we formulate Friedrichs' inequality for a plane domain and its connection with the inf–sup constant. We also derive a new inequality between the optimal constants figuring in Friedrichs' inequality. We call these constants in this paper Friedrichs' constants of the domain.

In Section 3 we first relate Friedrichs' constants of two simply connected plane domains using an estimation with the help of the function which maps one of the domains conformal onto the other. Then we make use of the known connection [19] between the inf–sup and Friedrichs' constant. If one of the domains is the unit disc, then we obtain a lower estimation for the inf–sup constant of the other domain in terms of its related Riemann mapping provided that the boundary of this domain is smooth enough. As a utilization of the deduced estimation we formulate a sufficient condition for the convergence of the inf–sup constants of a convergent domain sequence to the inf–sup constant of the limit domain.

2. Friedrichs' inequality and the inf–sup constant

Let Ω denote in this paper a plane domain which is further specified below. The value of the inf–sup constant $\beta_0(\Omega)$ is closely connected to the optimal constant in an inequality derived first by Friedrichs [7] between two real valued square integrable conjugate harmonic functions on Ω of which boundary is assumed to be piecewise smooth with finitely many corners. Friedrichs' inequality remains valid under weaker assumption for the boundary of the domain. It reads using the notation of this paper as follows.

Proposition 2.1. *(See H.S. Shapiro [18].) Let Ω be a bounded domain satisfying an interior cone condition and let $w_0 \in \Omega$. Let u and v be the real and imaginary parts of a square integrable complex analytic function on Ω subject to one of the normalizations*

$$\int_\Omega u\,dA = 0, \quad \textit{or} \tag{2}$$

$$u(w_0) = 0. \tag{3}$$

Then, for some finite constants Γ_Ω and $\tilde{\Gamma}_{\Omega,w_0}$, Friedrichs' inequality holds in either of the forms

$$\int_\Omega u^2\,dA \leqslant \Gamma_\Omega \int_\Omega v^2\,dA, \quad \textit{or} \tag{4}$$

$$\int_\Omega u^2\,dA \leqslant \tilde{\Gamma}_{\Omega,w_0} \int_\Omega v^2\,dA. \tag{5}$$

Let Γ_Ω and $\tilde{\Gamma}_{\Omega,w_0}$ denote the optimal constant in (4) and (5), that is the least positive number such that these inequalities are fulfilled for all pairs u and v. The value of Γ_Ω depends only on the shape of Ω but depends not on its size. The value of $\tilde{\Gamma}_{\Omega,w_0}$ depends additionally on the point w_0. There holds $\Gamma_\Omega \geqslant 1$ and $\tilde{\Gamma}_{\Omega,w_0} \geqslant 1$, moreover, for the unit disc $\mathbb{D}$ and $w_0 = 0$ the normalizations (2) and (3) are the same by the mean-value theorem and we obtain $\Gamma_{\mathbb{D}} = \tilde{\Gamma}_{\mathbb{D},0} = 1$, see [7].

The next lemma restates a known simple inequality between the two optimal constants Γ_Ω and $\tilde{\Gamma}_{\Omega,w_0}$ in the Friedrichs inequality [7], which was already utilized in [12] in case Ω is star-shaped with respect to the point w_0.

Lemma 2.2. *Let Ω be a bounded plane domain with boundary as in Proposition* 2.1. *We have*

$$\Gamma_\Omega \leqslant \tilde{\Gamma}_{\Omega,w_0} \tag{6}$$

for each $w_0 \in \Omega$.

Proof. If the harmonic function u satisfies the normalization (2), then there follows

$$\int_\Omega \big(u - u(w_0)\big)^2\,dA = \int_\Omega u^2\,dA - 2u(w_0)\int_\Omega u\,dA + u^2(w_0)|\Omega| \geqslant \int_\Omega u^2\,dA, \tag{7}$$

where $|\Omega|$ denotes the area of Ω. Now the harmonic functions $u - u(w_0)$ and v are also conjugate and $u - u(w_0)$ satisfies the normalization (3) at the point $w_0 \in \Omega$. Using (5) there follows from (7)

$$\frac{\int_\Omega u^2\,dA}{\int_\Omega v^2\,dA} \leqslant \frac{\int_\Omega (u - u(w_0))^2\,dA}{\int_\Omega v^2\,dA} \leqslant \tilde{\Gamma}_{\Omega,w_0},$$

which implies

$$\int_\Omega u^2\,dA \leqslant \tilde{\Gamma}_{\Omega,w_0}\int_\Omega v^2\,dA.$$

This gives (6), because Γ_Ω is the optimal constant in (4). $\square$

A reverse inequality to (6) holds if we extend it by a term connected with the point w_0.

Lemma 2.3. *Let Ω be a bounded plane domain with boundary as in Proposition* 2.1. *Let $|\Omega|$ denote the area of Ω and let $d = d(w_0, \partial\Omega)$ denote the distance of the point w_0 from the boundary $\partial\Omega$. For each $w_0 \in \Omega$ we have*

$$\tilde{\Gamma}_{\Omega,w_0} \leqslant \frac{|\Omega|}{d^2\pi}\Gamma_\Omega. \tag{8}$$

Proof. Let u and v be a pair of conjugate harmonic functions square integrable on the domain Ω such that $u(w_0) = 0$ for a fixed point $w_0 \in \Omega$. Define

$$\langle u\rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dA.$$

There follows $\int_\Omega (u - \langle u\rangle_\Omega)\,dA = 0$ and

$$\int_\Omega \bigl(u - \langle u\rangle_\Omega\bigr)^2\,dA = \int_\Omega u^2\,dA - |\Omega|\langle u\rangle_\Omega^2 = \int_\Omega u^2\,dA - \frac{1}{|\Omega|}\biggl(\int_\Omega u\,dA\biggr)^2. \tag{9}$$

Let $D_{w_0,r}$ denote a disc around the point w_0 with radius r fully contained in the domain Ω. By the mean-value theorem for harmonic functions and $u(w_0) = 0$ there follows $\int_{D_{w_0,r}} u\,dA = 0$ and hence we have $\int_\Omega u\,dA = \int_{\Omega\setminus D_{w_0,r}} u\,dA$. We estimate by the Cauchy–Schwartz inequality

$$\begin{aligned}\biggl(\int_\Omega u\,dA\biggr)^2 &= \biggl(\int_{\Omega\setminus D_{w_0,r}} u\,dA\biggr)^2 \leqslant \int_{\Omega\setminus D_{w_0,r}} 1\,dA \int_{\Omega\setminus D_{w_0,r}} u^2\,dA\\ &\leqslant |\Omega\setminus D_{w_0,r}|\cdot\int_{\Omega\setminus D_{w_0,r}} u^2\,dA \leqslant \bigl(|\Omega| - r^2\pi\bigr)\int_\Omega u^2\,dA.\end{aligned}$$

Using this estimation and (9) gives

$$\begin{aligned}\frac{\int_\Omega u^2\,dA}{\int_\Omega v^2\,dA} &= \frac{\int_\Omega (u - \langle u\rangle_\Omega)^2\,dA}{\int_\Omega v^2\,dA} + \frac{1}{|\Omega|}\cdot\frac{(\int_\Omega u\,dA)^2}{\int_\Omega v^2\,dA}\\ &\leqslant \frac{\int_\Omega (u - \langle u\rangle_\Omega)^2\,dA}{\int_\Omega v^2\,dA} + \biggl(1 - \frac{r^2\pi}{|\Omega|}\biggr)\frac{\int_\Omega u^2\,dA}{\int_\Omega v^2\,dA}\end{aligned}$$

for every $0 < r \leqslant d(w_0, \partial\Omega)$. We rearrange the latter inequality and we obtain

$$\frac{r^2\pi}{|\Omega|}\frac{\int_\Omega u^2\,dA}{\int_\Omega v^2\,dA} \leqslant \frac{\int_\Omega (u - \langle u\rangle_\Omega)^2\,dA}{\int_\Omega v^2\,dA},$$

which implies using Friedrichs inequality

$$\frac{r^2\pi}{|\Omega|}\frac{\int_\Omega u^2\,dA}{\int_\Omega v^2\,dA} \leqslant \Gamma_\Omega.$$

From this there follows

$$\int_\Omega u^2\,dA \leqslant \frac{|\Omega|}{r^2\pi}\Gamma_\Omega\int_\Omega v^2\,dA,$$

which gives (8) by setting $r = d(w_0, \partial\Omega)$. $\square$

The previous results mean that Friedrichs' constant Γ_Ω is comparable to the other Friedrichs' constant $\tilde{\Gamma}_{\Omega,w_0}$ with respect to a point. We summarize this in the following

Theorem 2.4. *Let Ω be a bounded plane domain with boundary as in Proposition* 2.1. *For each $w_0 \in \Omega$ there holds*

$$\Gamma_\Omega \leqslant \tilde{\Gamma}_{\Omega, w_0} \leqslant \frac{|\Omega|}{d^2 \pi} \Gamma_\Omega, \tag{10}$$

where $|\Omega|$ denotes the area of Ω and $d = d(w_0, \partial\Omega)$ denotes the distance of the point w_0 from the boundary $\partial\Omega$.

Example 2.5. We exemplify Theorem 2.4 by calculating Friedrichs' constant $\tilde{\Gamma}_{\mathbb{D}, w_0}$ for the unit disc. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be square integrable on the unit disc $\mathbb{D}$ and let be $u(z) = \operatorname{Re} f(z)$ and $v(z) = \operatorname{Im} f(z)$. If $u(w_0) = 0$ for a fixed $w_0 \in \mathbb{D}$, then there follows

$$-\operatorname{Re} c_0 = \operatorname{Re} \sum_{n=1}^{\infty} c_n w_0^n.$$

We estimate by the Cauchy–Schwartz inequality

$$|\operatorname{Re} c_0|^2 \leqslant \sum_{n=1}^{\infty} (n+1)|w_0|^{2n} \cdot \sum_{n=1}^{\infty} \frac{|c_n|^2}{n+1} = \left(\frac{1}{(1-|w_0|^2)^2} - 1 \right) \sum_{n=1}^{\infty} \frac{|c_n|^2}{n+1},$$

where the equality is valid for $c_n = (n+1)\overline{w_0}^n$. Using this we obtain

$$\frac{\int_{\mathbb{D}} u^2 \, dA}{\int_{\mathbb{D}} u^2 \, dA} = \frac{|\operatorname{Re} c_0|^2 + \sum_{n=1}^{\infty} \frac{|c_n|^2}{n+1}}{|\operatorname{Im} c_0|^2 + \sum_{n=1}^{\infty} \frac{|c_n|^2}{n+1}} \leqslant \frac{1}{(1-|w_0|^2)^2},$$

wherein equality holds for $f(z) = \frac{1}{(1-z\overline{w_0})^2}$. There follows

$$\tilde{\Gamma}_{\mathbb{D}, w_0} = \frac{1}{(1-|w_0|^2)^2}, \tag{11}$$

which we compare to (10). For the unit disc we have $d(w_0, \partial\Omega) = 1 - |w_0|$. Setting this and $\Gamma_{\mathbb{D}} = 1$ into (10) gives

$$1 \leqslant \tilde{\Gamma}_{\mathbb{D}, w_0} \leqslant \frac{|\mathbb{D}|}{(1-|w_0|)^2 \pi} \Gamma_{\mathbb{D}} = \frac{1}{(1-|w_0|)^2},$$

which constitutes a comparable result to the exact value (11):

$$\bigl(1-|w_0|\bigr)^2 \leqslant \bigl(1-|w_0|^2\bigr)^2 \leqslant 4\bigl(1-|w_0|\bigr)^2.$$

We have also reobtained $\Gamma_{\mathbb{D}} = \tilde{\Gamma}_{\mathbb{D},0} = 1$. $\square$

From the results in [23] it follows that Friedrichs' constant of a simply connected plane domain with Lipschitz continuous boundary is connected with the inf–sup constant of the same domain by the equality

$$\beta_0^2(\Omega) = \frac{1}{\Gamma_\Omega + 1}. \tag{12}$$

Using (12) along with the fact $\Gamma_\Omega \geqslant 1$ gives the upper estimation

$$\beta_0(\Omega) \leqslant \frac{1}{\sqrt{2}}, \tag{13}$$

where the equality holds for $\Omega = \mathbb{D}$. To obtain a more useful lower estimation for the inf–sup constant it suffices to have an upper estimation for Friedrichs' constant. Ref. [20] contains such an estimation for star shaped domains using a result for Friedrichs' constant from [12].

Connections between Friedrichs' constant and other domain specific optimal constants have been proved in [12].

3. Domain dependence and estimations

In this section conformal mappings are utilized to compare the Friedrichs' constants of two simply connected plane domains. We also derive estimations for the values of the constants in terms of the conformal mapping of the domain onto the unit disc. As a consequence of (12) there follow similar results for the inf–sup constant.

Theorem 3.1. *Let D be a simply connected plane domain with piecewise smooth boundary. Let η denote the bijective conformal mapping of the plane domain D onto Ω such that $\eta(z_0) = w_0$. Set $L = \sup_{\partial D} |\eta'| / \inf_{\partial D} |\eta'|$. If $0 < L < \infty$, then*

$$\frac{1}{L^2}\tilde{\Gamma}_{D,z_0} \leqslant \tilde{\Gamma}_{\Omega,w_0} \leqslant L^2\tilde{\Gamma}_{D,z_0} \tag{14}$$

follows for the constants $\tilde{\Gamma}_{D,z_0}$ and $\tilde{\Gamma}_{\Omega,w_0}$ from the inequality (5).

Proof. Let U be a harmonic function in Ω such that $U(w_0) = 0$ holds and it satisfies

$$\tilde{\Gamma}_{\Omega,w_0} = \frac{\int_\Omega U^2\,dA}{\int_\Omega V^2\,dA},$$

with its harmonic conjugate V. Define $u = U \circ \eta$, which is then harmonic in D, satisfies $u(z_0) = 0$ and has the conjugate $v = V \circ \eta$. We obtain

$$\tilde{\Gamma}_{\Omega,w_0} = \frac{\int_\Omega U^2\,dA}{\int_\Omega V^2\,dA} \leqslant \frac{\sup_D |\eta'|^2}{\inf_D |\eta'|^2}\frac{\int_\Omega u^2\,dA}{\int_\Omega v^2\,dA} \leqslant \frac{\sup_{\partial D} |\eta'|^2}{\inf_{\partial D} |\eta'|^2}\tilde{\Gamma}_{D,z_0},$$

which is the right-hand side of (14). The left-hand side can be similarly obtained if we calculate using the inverse of the mapping η. □

We have a similar relation between Friedrichs' constants of the two domains with respect to the normalization (2).

Theorem 3.2. *If D, Ω and L are as in Theorem* 3.1 *and Γ_D, Γ_Ω denote the optimal constants in Friedrichs' inequality on the domains D and Ω, respectively, then there follows*

$$\frac{1}{L^2}\Gamma_D \leqslant \Gamma_\Omega \leqslant L^2\Gamma_D. \tag{15}$$

Proof. Choose a pair of conjugate harmonic functions U and V on Ω such that U satisfies condition (2) and

$$\int_\Omega U^2\,dA \leqslant \Gamma_\Omega \int_\Omega V^2\,dA.$$

Let η be a conformal map of D onto Ω. Set $u = U \circ \eta$ and $v = V \circ \eta$. From (2) there follows $\int_\Omega u(z)|\eta'(z)|^2\,dA(z) = 0$, hence there is a point $z_* \in D$ such that $u(z_*) = 0$.

$$\Gamma_\Omega \geqslant \frac{\int_\Omega U^2\,dA}{\int_\Omega V^2\,dA} = \frac{\int_D u^2|\eta'|^2\,dA}{\int_D v^2|\eta'|^2\,dA} \geqslant \frac{\inf_D |\eta'|^2}{\sup_D |\eta'|^2}\cdot\frac{\int_D u^2\,dA}{\int_D v^2\,dA} = \frac{1}{L^2}\cdot\frac{\int_D u^2\,dA}{\int_D v^2\,dA}.$$

The inequality

$$\int_D u^2\,dA \leqslant L^2\Gamma_\Omega \int_D v^2\,dA$$

follows, which gives

$$\tilde{\Gamma}_{D,z_*} \leqslant L^2\Gamma_\Omega.$$

This implies using Lemma 2.2 the left-hand side of (15). The right-hand side of (15) follows similarly by using the inverse mapping. □

Using Theorem 3.2 and the correspondence between Friedrichs' constant and the inf–sup constant we obtain the following

Corollary 3.3. *Let D, Ω and L be as in Theorem* 3.2. *We have*

$$\frac{1}{L}\beta_0(D) \leqslant \beta_0(\Omega) \leqslant L\beta_0(D). \tag{16}$$

Proof. Use (12), (15) and $L \geqslant 1$ to prove the left-hand side inequality.

$$\beta_0^2(\Omega) = \frac{1}{1+\Gamma_\Omega} \geqslant \frac{1}{1+L^2\Gamma_D} \geqslant \frac{1}{L^2}\frac{1}{1+\Gamma_D} = \frac{1}{L^2}\beta_0^2(D).$$

We obtain the right-hand side inequality similar to this by changing the roles of the domains and using the inverse mapping. $\square$

Remark 3.4. The condition $0 < L < \infty$ in Theorem 3.1, Theorem 3.2 and Corollary 3.3 means a certain limitation for the regularity of conformal mapping between the domains and hence also for the utility of these results.

Example 3.5. We illustrate Theorem 3.2 on an example using the mapping properties of the function $\tilde{w} = \eta(w) = e^w$, which maps the rectangle

$$\Omega = \{w \in \mathbb{C}\colon \log r \leqslant \operatorname{Re} w \leqslant \log R,\ 0 \leqslant \operatorname{Im} w \leqslant \theta\},$$

where $0 < r < R$, $|\theta| < 2\pi$, conformally onto the annular sector

$$\tilde{\Omega} = \left\{\tilde{w} \in \mathbb{C}\colon r \leqslant |\tilde{w}| \leqslant R,\ 0 \leqslant \arg w \leqslant \theta\right\},$$

see [16]. We have $\eta'(w) = e^w$, which gives $|\eta'(w)|^2 = |e^w|^2 = e^{2\operatorname{Re} w}$ and

$$r^2 \leqslant \left|\eta'(w)\right|^2 \leqslant R^2 \quad \text{for } w \in \Omega.$$

By Theorem 3.2 and the corollary thereafter, we obtain

$$\Gamma_{\tilde{\Omega}} \leqslant \Gamma_\Omega \frac{R^2}{r^2} \quad \text{and} \quad \beta_0(\tilde{\Omega}) \geqslant \beta_0(\Omega)\frac{r}{R}. \tag{17}$$

Now we use a lower estimate for the inf–sup constant of the rectangle Ω given in [20]: $\beta_0(\Omega) \geqslant \frac{1}{M}\sin\frac{\pi}{8}$, where $M \geqslant 1$ is the ratio between the sides of the rectangle. In this case we have either

$$M = \frac{\log\frac{R}{r}}{\theta} \quad \text{or} \quad M = \frac{\theta}{\log\frac{R}{r}},$$

depending on the values of the parameter r, R and θ. If we substitute these expressions into (17), then we obtain a lower estimation of the inf–sup constant $\beta_0(\tilde{\Omega})$ of an annular sector depending only on its geometric properties.

If we choose $D = \mathbb{D}$ and $z_0 = 0$, then by Theorems 3.1 and 3.2 we obtain estimations of the constants in terms of the conformal mapping of Ω onto the unit disc $\mathbb{D}$.

Corollary 3.6. *Let R be the conformal map of the plane domain Ω onto the unit disc $\mathbb{D}$ such that $R(w_0) = 0$ and $|R'|$ has a positive lower and upper bound on $\partial\Omega$. Set $L = \sup_{\partial\Omega}|R'|^2/\inf_{\partial\Omega}|R'|^2$. Then we have*

$$1 \leqslant \tilde{\Gamma}_{\Omega,w_0} \leqslant L, \tag{18}$$

$$1 \leqslant \Gamma_\Omega \leqslant L, \tag{19}$$

and

$$\frac{1}{\sqrt{2}L} \leqslant \beta_0(\Omega) \leqslant \frac{1}{\sqrt{2}}. \tag{20}$$

The equalities hold for $\Omega = \mathbb{D}$ and $R(w) = w$.

Remark 3.7. The preceding Corollary 3.6 is applicable only for domains with enough smooth boundary in order to have the modulus of the derivative of the corresponding mapping function bounded away from zero and from infinity on the boundary of the domain. By a theorem of Kellogg [13], if Ω is a domain bounded by a smooth closed Jordan curve for which the angle of inclination $\phi(s)$ of the tangent to the real axis, as a function of the arc length s of $\partial\Omega$, satisfies a Hölder condition, then the conformal mapping fulfills the needed condition in Corollary 3.6.

Example 3.8. As an illustration of Corollary 3.6 we can consider domains $\Omega = \eta(\mathbb{D})$ for which

$$\eta(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

such that $\sum_{n=2}^{\infty} n|a_n| \leqslant \varepsilon < 1$. Such domains are schlicht and nearly circular in the sense that

$$\pi \leqslant |\Omega| \leqslant \pi(1+\varepsilon) \quad \text{and} \quad 2\pi \leqslant |\partial\Omega| \leqslant 2\pi(1+\varepsilon),$$

where $|\Omega|$ denotes the area of Ω and $|\partial\Omega|$ denotes the length of the closed curve $\partial\Omega$. There follows $0 < 1-\varepsilon \leqslant |\eta'| \leqslant 1+\varepsilon$ and

$$\beta_0(\Omega) \geqslant \frac{1}{\sqrt{2}} \cdot \frac{1-\varepsilon}{1+\varepsilon}.$$

Using $\beta_0(\mathbb{D}) = \frac{1}{\sqrt{2}}$ and $\varepsilon < 1$ we obtain

$$\bigl|\beta_0(\mathbb{D}) - \beta_0(\Omega)\bigr| \leqslant \frac{1}{\sqrt{2}} \frac{2\varepsilon}{1+\varepsilon} \leqslant \sqrt{2}\varepsilon,$$

which means that the inf–sup constant of a nearly circular domain is near to the inf–sup constant of the disc.

As a direct consequence of Corollary 3.3 we obtain

Corollary 3.9. *Let D, Ω, η be as in Corollary* 3.3. *If $|\eta' - 1| \leqslant \varepsilon < 1$ in the closure of D, then we have*

$$\bigl|\beta_0(\Omega) - \beta_0(D)\bigr| \leqslant \frac{\sqrt{2}\varepsilon}{1-\varepsilon}. \tag{21}$$

Proof. The inequality (16) can be formulated as

$$\bigl|\beta_0(\Omega) - \beta_0(D)\bigr| \leqslant (L-1)\beta_0(D),$$

because for $L \geqslant 1$ the estimation $1 - L \leqslant \frac{1}{L} - 1$ holds. Now, from $|\eta' - 1| \leqslant \varepsilon < 1$ there follows $0 < 1-\varepsilon \leqslant |\eta'| \leqslant 1+\varepsilon$ and

$$L = \frac{\sup_D |\eta'|}{\inf_D |\eta'|} \leqslant \frac{1+\varepsilon}{1-\varepsilon},$$

which gives $L - 1 \leqslant \frac{2\varepsilon}{1-\varepsilon}$ and

$$\bigl|\beta_0(\Omega) - \beta_0(D)\bigr| \leqslant \frac{2\varepsilon}{1-\varepsilon}\beta_0(D).$$

Substituting (13) into the latter inequality implies (21). $\square$

Let the sequence of domains $\Omega_n = g_n(\mathbb{D})$, $n = 1, 2, \ldots,$ tend to the domain $\Omega = g(\mathbb{D})$ in the sense that for their corresponding conformal mappings g_n, g one has the following: for every $0 < \varepsilon < 1$ there exists a natural number N such that

$$\sup_D \bigl|g_n' - g'\bigr| < \varepsilon, \tag{22}$$

whenever $n > N$. Then by Corollary 3.9 there follows

$$\lim_{n\to\infty} \beta_0(\Omega_n) = \beta_0(\Omega),$$

i.e. one has the convergence of the inf–sup constants. Therefore we have obtained a sufficient condition for the convergence of the inf–sup constants of a convergent domain sequence to the inf–sup constant of the limit domain.

If one has a condition weaker than (22), then the convergence of the inf–sup constants cannot be guaranteed [25]. Consider the functions

$$g_{(m,\alpha)}(z) = z - \frac{c}{m^{\alpha+1}} z^m \tag{23}$$

($c \in \mathbb{R}$, $\alpha \geqslant 0$, $m > 1$ and integer), which are univalent in the unit disc $\mathbb{D}$ for $|c| \leqslant m^\alpha$. Set $\Omega_{(m,\alpha)} = g_{(m,\alpha)}(\mathbb{D})$. Such a domain is a quadrature domain and its inf–sup constant can be computed as an eigenvalue of a finite matrix, [25].

$$\beta_0^2(\Omega_{(m,\alpha)}) = \begin{cases} \frac{1}{2}\bigl(1 - \frac{m+1}{2} \cdot \frac{|c|}{m^{\alpha+1}}\bigr) & \text{for } m \text{ odd}, \\ \frac{1}{2}\bigl(1 - \sqrt{\frac{m}{2}(\frac{m}{2}+1)}\frac{|c|}{m^{\alpha+1}}\bigr) & \text{for } m \text{ even}. \end{cases} \tag{24}$$

We compute for $\alpha := 0$ and $0 < |c| < 1$

$$\|g_m - g\|_0^2 = \frac{|c|^2}{m^2} \int_D z^m \bar{z}^m \, dx\, dy = \frac{\pi |c|^2}{m^2(m+1)},$$

$$\max_{z\in D} \big|g_m(z) - g(z)\big| = \frac{|c|}{m} \max_{z\in D} \big|z^m\big| = \frac{|c|}{m},$$

$$\big\|g'_m - g'\big\|_0^2 = \frac{\pi |c|^2}{m},$$

$$\max_{z\in D} \big|g'_m(z) - g'(z)\big| = |c| \max_{z\in D} |z|^{m-1} = |c|,$$

where we have set $g_m := g_{(m,0)}$ and $g(z) = z$. These equalities show that $\lim_{m\to\infty} g_m = g$ in the L_2 and maximum norm on D, further $\lim_{m\to\infty} g'_m = g'$ is valid in the L_2 norm but not in the maximum norm. In this sense we have a sequence of smoothly bounded plane domains Ω_m, $m = 1, 2, \ldots$ converging to the unit disc $\mathbb{D}$. The limit of the inf–sup constants (24) of the domains is however

$$\lim_{m\to\infty} \beta_0^2(\Omega_m) = \frac{1}{2} - \frac{|c|}{4} < \frac{1}{2} = \beta_0^2(\mathbb{D}).$$

Remark 3.10. Friedrichs' constant – and hence also the inf–sup constant – is connected to other domain specific optimal constants in important inequalities. For example we have from [12] $K_\Omega = 2 + 2\Gamma_\Omega$ for the optimal constant in Korn's second inequality on the plane domain Ω. All the derived estimations in this paper are therefore also valid for these other constants.

4. Concluding remarks

In this paper we have obtained results for the domain dependence of important domain specific constants, especially of the inf–sup stability constant. The conformal mapping approach is usable for domains with enough smooth boundaries and it allows to give a criterion for the continuous dependence of the constants on the domain. In case the mapping function is singular on the boundary, the used integral estimations are ineffective. Nevertheless, the last example shows that the continuous domain dependence of the inf–sup constant fails also in the family of smoothly bounded domains if the given criterion is violated. Hence the singularity of the mapping function on the domain boundary is not the only reason for the failure of the continuous domain dependence of the examined constants.

References

[1] I. Babuška, A.K. Aziz, Survey lectures on the mathematical foundations of the finite element method, in: A.K. Aziz (Ed.), The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Academic Press, New York, 1972.

[2] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.

[3] E.V. Chizhonkov, On the constant in the LBB condition for ring domains, Report No. 9537, Dept. Math., Univ. of Nijmegen, Sept. 1995.

[4] E.V. Chizhonkov, M.A. Olshanskii, On the domain geometry dependence of the LBB condition, Math. Model. Numer. Anal. 34 (5) (2000) 935–951.

[5] G. Duvaut, J.L. Lions, Inequalities in Mechanics and Physics, Comprehensive Stud. Math., vol. 219, Springer, 1976.

[6] M. Dobrowolski, On the LBB constant on stretched domains, Math. Nachr. 254–255 (2003) 64–67.

[7] K. Friedrichs, On certain inequalities and characteristic value problems for analytic functions and for functions of two variables, Trans. Amer. Math. Soc. 41 (1937) 321–364.

[8] K. Friedrichs, On the boundary-value problems of the theory of elasticity and Korn's inequality, Ann. of Math. 48 (1947) 441–471.

[9] P.R. Garabedian, A partial differential equation arising in conformal mapping, Pacific J. Math. 1 (1951) 485–524.

[10] V. Girault, P.A. Raviart, Finite Element Methods for Navier–Stokes Equations, Springer, New York, 1986.

[11] C.O. Horgan, Korn's inequalities and their applications in continuum mechanics, SIAM Rev. 37 (4) (1995) 491–511.

[12] C.O. Horgan, L.E. Payne, On inequalities of Korn, Friedrichs and Babuška–Aziz, Arch. Ration. Mech. Anal. 82 (1983) 165–179.

[13] O.D. Kellogg, On the derivatives of harmonic functions on the boundary, Trans. Amer. Math. Soc. 33 (2) (1931) 486–510.

[14] Y. Maday, D. Meiron, A.T. Patera, E.M. Ronquist, Analysis of iterative methods for the steady and unsteady Stokes problem: application to spectral element discretizations, SIAM J. Sci. Comput. 14 (1993) 310–337.

[15] J. Nečas, Equations aux Dérivées Partielles, Presses le l'Université de Montreal, 1965.

[16] Z. Nehari, Conformal Mapping, McGraw–Hill Book Company, Inc., New York, 1952.

[17] M. Sakai, Quadrature Domains, Lecture Notes in Math., vol. 934, Springer-Verlag, 1982, pp. 287–331.

[18] H.S. Shapiro, On some Fourier and distribution-theoretic methods in approximation theory, in: W. Cheney, et al. (Eds.), Approximation Theory III, Proc. Conf., Austin, Texas, 1980, Academic Press, San Diego, 1980, pp. 87–124.

[19] G. Stoyan, Towards discrete Velte decompositions and narrow bounds for inf–sup constants, Comput. Math. Appl. 38 (7–8) (1999) 243–261.

[20] G. Stoyan, Iterative Stokes solvers in the harmonic Velte subspace, Computing 67 (2000) 13–33.

[21] R. Temam, Navier–Stokes Equations, North-Holland, New York, 1977.

[22] W. Velte, On optimal constants in some inequalities, in: J.G. Heywood, et al. (Eds.), The Navier–Stokes Equations, Theory and Numerical Methods, in: Lecture Notes in Math., vol. 1431, Springer-Verlag, Berlin, 1990, pp. 158–168.

[23] W. Velte, On inequalities of Friedrichs and Babuška–Aziz, Meccanica 31 (1996) 589–596.

[24] S. Zimmer, Rand-Druckkorrektur für die Stokes-Gleichung, Thesis, Tech. Univ. München, 1996.

[25] S. Zsuppan, On the spectrum of the Schur complement of the Stokes operator via conformal mapping, Methods Appl. Anal. 11 (1) (2004) 133–154.